Fuzzy-Model-Based Exponentially Stabilizing Perturbed Nonlinear Systems in the Presence of Modeling Error

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***Abstract*—This paper presents a systematic design procedure of fuzzy controllers for exponentially stabilizing affine nonlinear systems, which are subject to vanishing perturbations. The modeling error, as a key factor in the validity of the stability analysis is also taken into consideration. A unified framework is presented, dealing with the modeling error as the system perturbation. It is shown that the constraints imposed on the modeling error are always satisfied for affine nonlinear systems, when they are modeled by Takagi-Sugeno fuzzy model through currently available methods. The constraints required to guarantee the exponential stability of the original nonlinear system and design the controller are transformed into LMIs. Finally, the way to utilize the presented method for a stabilizing problem is demonstrated using two examples.**

I. Introduction

CONTROL of nonlinear dynamical systems, because of their structural complicacy and behavioral variety, is still a very challenging area in the control systems engineering. Designing a controller to stabilize and improve the performance of these systems, usually involves a relatively strong mathematical and engineering background. So, the theories and methods developed to analyze and design nonlinear control systems are hardly acceptable in industry.

According to the problems mentioned above, developing a systematic design method which is easily understandable and applicable to a large family of nonlinear systems, is an important research issue in modern control engineering. Recently, the fuzzy-model-based approaches to design controller for nonlinear systems has taken a large step toward achieving this objective. In most of these approaches, the original nonlinear system is first replaced by its Takagi-Sugeno (TS) fuzzy approximation, which is a weighted sum of linear subsystems. The controller structure is then chosen accordingly as a weighted sum of state feedback controllers, with the same weights as the TS model weights. This structure for controller is known as parallel distributed compensator (PDC). The control design objective is then, determining feedback gains of the PDC such that the origin of the fuzzy closed loop system, i.e. the system consisted of the TS model and PDC, is stable or some performance criteria are satisfied. Applying quadratic Lyapunov function to analyze the stability of this closed loop system provides the ability to present the stability conditions in the form of LMIs. Since very efficient numerical methods are currently available for solving LMIs ([1]), the controller gains can be obtained automatically via solving these LMIs. Consequently, the controller design procedure of the original nonlinear system is done semi-automatically.

The idea described above, is the basis of many fuzzy-model-based methods developed so far to stabilize nonlinear systems, e.g. in [2]-[5]. However, a very important point has been disregarded in the abovementioned procedure. The modeling error, which is the error between the original nonlinear system and its TS fuzzy approximation, has not been considered in the stability analysis. In the presence of a non-zero error, the stability analysis of the closed loop fuzzy system has no theoretical worth in guaranteeing the stability of the original closed loop system, i.e. the system consisted of the PDC and the original nonlinear system. In spite of the undeniable importance of this subject, only a few researches have ever consciously considered the modeling error in the stability analysis, as it is noticed in [6] and also recalled in [7] that "*there are few results on whether it is possible that the fuzzy controller, which is designed to stabilize the TS fuzzy model, can also stabilize the original nonlinear system, and furthermore if it is possible, how to design the fuzzy controller to achieve such stabilization of the original nonlinear system.*" Introducing such a point of view has motivated some researches and led to some valuable results. In [7], a preliminary answer to this question has been presented that considers both the possibility and the way of realization of this approach. Investigating the possibility, [7] states that "*the class of nonlinear systems whose stabilization can be solved by available fuzzy control approach based on TS models is affine nonlinear systems*" and as the way of realization of this approach it states that "*the stabilization problem of an affine nonlinear system can be solved as a robust stabilization problem of its TS fuzzy approximator with the approximation error as the uncertainty bounds.*"

This paper investigates the problem of exponentially stabilizing perturbed affine nonlinear systems. The perturbation is supposed to be vanishing at the origin, and can be caused by some uncertainties, aging, or modeling error. These are some unavoidable facts in realistic problems which can highly affect the stability and performance of the system, and a reliable design method must be robust against

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these perturbations.

As implicitly noted above, one of the sources of the system perturbation can also be the modeling error. So, using the conventional framework for analyzing the stability of perturbed nonlinear systems, as introduced in [8], we present the modeling error as a vanishing perturbation, satisfying some norm boundedness conditions. Based on the results in [3] and [7], we discuss that this presentation is always possible and the conditions are always satisfied for affine nonlinear systems. Such a presentation provides a uniform framework which deals equivalently with modeling error and the perturbations of the original nonlinear system.

Based on the described presentation of and assumptions on the modeling error and system perturbations, the required constraints, guaranteeing the exponential stability of the original closed loop system are derived in the form of LMIs. Finally, the way of utilizing the presented method, is illustrated via two examples.

II. Problem Definition and Error Presentation

A. Problem Definition

Consider the following perturbed affine nonlinear system:

$$\dot{x} = f(x) + g(x)u = \left(\hat{f}(x) + \delta_{f1}(x)\right) + \left(\hat{g}(x) + \delta_{g1}(x)\right)u \qquad (1)$$

Where $f(0) = 0$, $x \in R^n$ is the state vector, $u \in R^m$ is the input vector, and $\delta_{f1}(x)$ and $\delta_{g1}(x)$ are perturbations. The perturbation term is supposed to be vanishing at the origin, i.e. $\delta_{f1}(0) = 0$. The nominal system is obtained by ignoring the perturbation terms in (1) and can be modeled by the standard TS fuzzy approximation as:

$$\begin{aligned}\dot{x} &= \hat{f}(x) + \hat{g}(x)u \\ &= \sum_{i=1}^{r} h_i(x)(A_i x + B_i u) + \delta_{f2}(x) + \delta_{g2}(x)u\end{aligned} \qquad (2)$$

where $\delta_{f2}(x)$ and $\delta_{g2}(x)$ present the modeling error, and $\delta_{f2}(0) = 0$. r is the number of linear subsystems and $h_i(x)$ is the corresponding weight of the i^{th} subsystem, satisfying:

$$\sum_i h_i(x) = 1, \qquad h_i(x) \ge 0, \qquad i = 1,2,\ldots,r$$

With this unified type of presenting the modeling error and system perturbations, we can combine (1) and (2) and describe the original perturbed system as:

$$\dot{x} = f(x) + g(x)u = \sum_{i=1}^{r} h_i(x)(A_i x + B_i u) + \delta_f(x) + \delta_g(x)u \qquad (3)$$

where $\delta_f(x) = \delta_{f1}(x) + \delta_{f2}(x)$, $\delta_g(x) = \delta_{g1}(x) + \delta_{g2}(x)$. We assume that the perturbation term $\delta_f(x)$ satisfies the conventional linear growth bound ([8]) with 2-norm, and $\delta_g(x)$ has bounded 2-norm, as following:

$$\left\|\delta_f(x)\right\|_2 \le \gamma_f \left\|x\right\|_2, \qquad \left\|\delta_g(x)\right\|_2 \le \beta_g \qquad (4)$$

Let the controller be the commonly used PDC structure as

$$u = -\sum_{i=1}^{r} h_i(x) F_i x \qquad (5)$$

where F_i is the feedback gain corresponding to the i^{th} subsystem. If we apply this controller to (3), we obtain the following closed loop system:

$$\begin{aligned}\dot{x} &= \sum_i h_i(x)\left(A_i x - B_i \sum_j h_j(x) F_j x\right) + \delta_f(x) - \delta_g(x) \\ &\quad \times \sum_j h_j(x) F_j x = \sum_i \sum_j h_i(x) h_j(x) G_{ij}(x) x + \delta_f(x)\end{aligned} \qquad (6)$$

Where,

$$G_{ij}(x) = A_i - B_i F_j - \delta_g(x) F_j \qquad (7)$$

and $\sum_i$ means $\sum_{i=1}^{r}$ in allover the paper.

B. Discussion of Possibility

In the previous subsection we presented the modeling error as system perturbations, and assumed some special norm boundedness properties (4) on these perturbation terms. But, a question might be asked here that, whether it is possible to do such presentation satisfying (4) or not. In this subsection, we try to find an answer for this question.

It is shown in [7] that affine nonlinear systems can always be approximated by standard TS fuzzy model plus the modeling error terms as:

$$\begin{aligned}\delta_f(x) &= \varepsilon_A(x)x = \sum_i h_i(x)\varepsilon_{A_i}(x)x \\ \delta_g(x) &= \varepsilon_B(x) = \sum_i h_i(x)\varepsilon_{B_i}(x)\end{aligned} \qquad (8)$$

which satisfy the following bounds with any $\varepsilon > 0$:

$$\left\|\varepsilon_{A_i}(x)\right\|_\infty < \varepsilon, \qquad \left\|\varepsilon_{B_i}(x)\right\|_\infty < \varepsilon, \qquad i = 1,\ldots,r \qquad (9)$$

Now we show that the error terms (8) satisfying (9), will also satisfy the constraints (4). First consider the term $\delta_f(x)$ in (8). According to some matrix norm inequalities found in [9], [10] or some other text books on linear algebra, we can easily write:

$$\begin{aligned}\left\|\delta_f(x)\right\|_2 &= \left\|\sum_i h_i(x)\varepsilon_{A_i}(x)x\right\|_2 \le \sum_i h_i(x)\left\|\varepsilon_{A_i}(x)\right\|_2 \left\|x\right\|_2 \\ &\le \sum_i h_i(x)\sqrt{n}\left\|\varepsilon_{A_i}(x)\right\|_\infty \left\|x\right\|_2 < \sum_i h_i(x)\sqrt{n}\varepsilon\left\|x\right\|_2 = \sqrt{n}\varepsilon\left\|x\right\|_2\end{aligned}$$

So, it can be seen that the first inequality in (4) is satisfied with $\gamma_f = \sqrt{n}\varepsilon$. Similarly, for $\delta_g(x)$ we can write:

$$\begin{aligned}\left\|\delta_g(x)\right\|_2 &= \left\|\sum_i h_i(x)\varepsilon_{B_i}(x)\right\|_2 \le \sum_i h_i(x)\left\|\varepsilon_{A_i}(x)\right\|_2 \\ &\le \sum_i h_i(x)\sqrt{n}\left\|\varepsilon_{B_i}(x)\right\|_\infty < \sum_i h_i(x)\sqrt{n}\varepsilon = \sqrt{n}\varepsilon\end{aligned}$$

Consequently, the second inequality in (4) is also satisfied with $\beta_g = \sqrt{n}\varepsilon$.

As another method of modeling, we consider the method using the sector nonlinearity approach described in [2]. It seems that this method of modeling was not investigated in [7], since this method can locally (or globally) *represent* rather than *approximate* the original nonlinear system. In other words, modeling via this method is exact, i.e. the modeling error equals to zero. So, In this case, it is clear that the stability analysis of the fuzzy closed loop system is also valid locally (or globally) for the original closed loop system. However, in order to decrease the computational effort required to solve the LMIs, a rule reduction procedure is proposed in [3]. This rule reduction procedure produces a modeling error, which is then converted to norm-bounded model uncertainties. Here we show that the bounded modeling error obtained in [3] can also be presented as in (2) and always satisfies the constraints (4).

Referring to [3], it can be easily seen that the modeling error terms are presented as:

$$\delta_f(x(t)) = \sum_i h_i(x(t)) D_{ai}\Delta_{ai}(t)E_{ai}x(t)$$

$$\delta_g(x(t)) = \sum_i h_i(x(t)) D_{bi}\Delta_{bi}(t)E_{bi} \tag{10}$$

where D_{ai}, D_{bi}, E_{ai}, and E_{bi} are known matrices, and $\Delta_{ai}(t)$ and $\Delta_{bi}(t)$ satisfy the upper bounds:

$$\|\Delta_{ai}(t)\|_2 < \frac{1}{\rho_{ai}}, \quad \|\Delta_{bi}(t)\|_2 < \frac{1}{\rho_{bi}}, \quad i = 1,\ldots,r \tag{11}$$

For the perturbation term $\delta_f(x)$ we can write:

$$\|\delta_f(x)\|_2 \le \sum_i h_i(x)\|D_{ai}\|_2\|\Delta_{ai}(t)\|_2\|E_{ai}\|_2\|x\|_2$$

The worst case is obtained when all *r* fuzzy rules are reduced to just one rule. In this case, by considering the definitions (12) found in [3], $\|D_{ai}\|_2$, $\|\Delta_{ai}(t)\|_2$, and $\|E_{ai}\|_2$ take their maximum values as $\|D_{a1}\|_2 = \sqrt{n}$, $\|\Delta_{a1}(t)\|_2 \le 1/\rho_{a1}$, and $\|E_{a1}\|_2 = \sqrt{n}$.

$$D_{a1} = diag(L_n,\ldots L_n) \in R^{n\times n^2},\ L_n = [\underbrace{1,\ldots,1}_{n}] \in R^{1\times n}$$

$$D_{b1} = diag(L_m,\ldots L_m) \in R^{n\times nm},\ L_n = [\underbrace{1,\ldots,1}_{m}] \in R^{1\times m}$$

$$E_{a1} = [I_n,\ldots I_n]^T \in R^{n^2\times n},\ E_{b1} = [I_m,\ldots I_m]^T \in R^{nm\times m} \tag{12}$$

Based on these worst case bounds, we can write:

$$\|\delta_f(x)\|_2 \le \sum_i h_i(x)\|D_{a1}\|_2\|\Delta_{a1}(t)\|_2\|E_{a1}\|_2\|x\|_2 = \frac{n}{\rho_{a1}}\|x\|_2$$

Which shows that the first inequality in (4) is satisfied with $\gamma_f = \frac{n}{\rho_{a1}}$. Similarly, using the definitions in (12), we can write the following inequalities for $\delta_g(x)$:

$$\|\delta_g(x)\|_2 \le \sum_i h_i(x)\|D_{bi}\|_2\|\Delta_{bi}(t)\|_2\|E_{bi}\|_2$$

$$\le \sum_i h_i(x)\|D_{b1}\|_2\|\Delta_{b1}(t)\|_2\|E_{b1}\|_2 = \frac{\sqrt{mn}}{\rho_{b1}}$$

Consequently, the second inequality in (4) is also satisfied with $\beta_g = \frac{\sqrt{mn}}{\rho_{b1}}$.

Now, we can conclude that the presentation of modeling error as in (2) while satisfying the constraints (4), is always possible for affine nonlinear systems which are modeled by at least two currently available methods.

III. Stability Analysis and Controller Design

In this section, using a common quadratic Lyapunov function, we analyze the exponential stability of the closed loop system (6), and transform the required constraints for stability into LMIs.

Before starting the stability analysis, and to make ease of reference, we first recall a theorem from [8].

Consider the nonlinear system $\dot{x} = f(t,x)$, where $f:[0,\infty)\times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,\infty)\times D$, and $D \subset R^n$ is a domain that contains the origin $x = 0$. Let $x = 0$ be an equilibrium point for this system, and $V:[0,\infty)\times D \to R$ be a continuously differentiable function such that

$$c_1\|x\|^a \le V(t,x) \le c_2\|x\|^a \tag{13}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_3\|x\|^a \tag{14}$$

for $\forall t \ge 0$ and $\forall x \in D$, where c_1, c_2, c_3, and a are positive constants. Then the theorem in [8] states that $x = 0$ is exponentially stable, and if the assumptions hold globally, then $x = 0$ is globally exponentially stable.

Also, we recall the following inequality from [2]

$$\sum_i h_i^2(x) - \frac{1}{s-1}\sum_i\sum_{i<j} 2h_i(x)h_j(x) \ge 0 \tag{15}$$

when the number of rules that fire for all t is less than or equal to s, where $1 < s \le r$.

According to the theorem and the inequality introduced above, and the problem defined in the previous section, the following theorem gives the design constraints to guarantee the exponential stability of the original closed loop system.

Theorem: The matrix gains of the PDC (5), which exponentially stabilizes the origin $x = 0$ of the perturbed nonlinear system (1), being presented as (3), and satisfying (4), are obtained by solving the LMIs (16)-(18). In these LMIs, the matrices *X*, Y_0, and M_is and the scalars σ_1 and σ_2 are variables to be determined, and $M_i = F_i X$. $\kappa_f = 1$ ($\kappa_g = 1$) when $\delta_f(x) \ne 0$ ($\delta_g(x) \ne 0$), and $\kappa_f = 0$ ($\kappa_g = 1$) otherwise. When $\delta_f(x) = 0$ ($\delta_g(x) = 0$), the corresponding rows and columns in (17) and (18), signified by dashed (dotted) lines, are omitted.

$$X > 0, \quad Y_0 > 0, \quad \sigma_1 > 0, \quad \sigma_2 > 0 \tag{16}$$

$$\begin{bmatrix} \begin{pmatrix} XA_i^T + A_i X - M_i^T B_i^T - B_i M_i \\ +(s-1)Y_0 + \kappa_f \sigma_1 I + \kappa_g \sigma_2 I \end{pmatrix} & X & M_i^T \\ X & -\dfrac{\sigma_1}{\gamma_f^2} & 0 \\ M_i & 0 & -\dfrac{\sigma_2}{\beta_g^2} \end{bmatrix} \le 0$$

$$\forall i \tag{17}$$

$$\begin{bmatrix} \begin{pmatrix} XA_i^T + A_i X - M_j^T B_i^T - B_i M_j \\ +XA_j^T + A_j X - M_i^T B_j^T - B_j M_i \\ -2Y_0 + 2\kappa_f \sigma_1 I + 2\kappa_g \sigma_2 I \end{pmatrix} & X & M_i^T & M_j^T \\ X & -\dfrac{\sigma_1}{2\gamma_f^2} & 0 & 0 \\ M_i & 0 & -\dfrac{\sigma_2}{\beta_g^2} & 0 \\ M_j & 0 & 0 & -\dfrac{\sigma_2}{\beta_g^2} \end{bmatrix} \le 0$$

$$\forall i < j \ \text{ s.t. } h_i \cap h_j \neq \phi \tag{18}$$

Furthermore, if the modeling (3) is valid globally, and the assumptions (4) hold globally too, then the origin $x = 0$ is globally exponentially stable.

Proof: Consider the candidate for Lyapunov function as $V(x) = x^T P x$ where $P = P^T > 0$. This function obviously satisfies (13) with 2-norm and $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$, and $a = 2$. So, in order to satisfy (14), we must have

$$\dot{V}(x) \le -c_3 \|x\|_2^2 = -c_3 x^T x, \quad c_3 > 0$$

If we compute the derivative of the Lyapunov function along the trajectories of the closed loop system (6), we have

$$\dot{V}(x) + c_3 x^T x = \sum_i \sum_j h_i(x) h_j(x) x^T \Big(G_{ij}^T(x) P + P G_{ij}(x) + c_3 I \Big) x + \delta_f^T(x) P x + x^T P \delta_f(x) \tag{19}$$

For the second term in (19) and employing (4), we can write

$$\delta_f^T(x) P x + x^T P \delta_f(x) = -\left(\frac{1}{\sigma_1} \delta_f(x) - P x \right)^T \Big(\delta_f(x) - \sigma_1 P x \Big) + \frac{1}{\sigma_1} \delta_f^T(x) \delta_f(x) + \sigma_1 x^T P^2 x \le \frac{1}{\sigma_1} \delta_f^T(x) \delta_f(x) + \sigma_1 x^T P^2 x$$

$$\le \frac{1}{\sigma_1} \gamma_f^2 I \|x\|_2^2 + \sigma_1 x^T P^2 x = x^T \left(\frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \right) x$$

Substituting this inequality into (19), we have

$$\dot{V}(x) + c_3 x^T x \le \sum_i \sum_j h_i(x) h_j(x) x^T \Big(G_{ij}^T(x) P + P G_{ij}(x) + c_3 I + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \Big) x = \sum_i h_i^2(x) x^T \Big(G_{ii}^T(x) P + P G_{ii}(x) + c_3 I + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \Big) x$$
$$+ \sum_i \sum_{i<j} 2 h_i(x) h_j(x) x^T \left(\left(\frac{G_{ij}(x) + G_{ji}(x)}{2} \right)^T P + P \left(\frac{G_{ij}(x) + G_{ji}(x)}{2} \right) + c_3 I + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \right) x \tag{20}$$

Now consider a positive semi-definite matrix Q such that

$$\left(\frac{G_{ij}(x) + G_{ji}(x)}{2} \right)^T P + P \left(\frac{G_{ij}(x) + G_{ji}(x)}{2} \right) + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \le Q$$
$$Q \ge 0 \quad i < j, \quad s.t. \ h_i \cap h_j \neq \phi \tag{21}$$

Substituting this inequality in (20), we have

$$\dot{V}(x) + c_3 x^T x \le \sum_i h_i^2(x) x^T \left(G_{ii}^T(x) P + P G_{ii}(x) + c_3 I + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \right) x + \sum_i \sum_{i<j} 2 h_i(x) h_j(x) x^T (Q + c_3 I) x$$

which can be written using the inequality (15) as

$$\dot{V}(x) + c_3 x^T x \le \sum_i h_i^2(x) x^T \left(G_{ii}^T(x) P + P G_{ii}(x) + c_3 I + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \right) x + (s-1) \sum_i h_i^2(x) x^T (Q + c_3 I) x = \sum_i h_i^2(x) x^T$$
$$\times \left(G_{ii}^T(x) P + P G_{ii}(x) + s c_3 I + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 + (s-1) Q \right) x$$

Now, in order to satisfy (14), it is sufficient to have:

$$G_{ii}^T(x) P + P G_{ii}(x) + s c_3 I + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 + (s-1) Q \le 0 \tag{22}$$

In (21), we considered Q to be a positive semi-definite matrix. If we make this condition a bit stricter and consider Q to be a positive definite matrix ($Q > 0$), then by choosing $0 < c_3 \le (s-1) \lambda_{\min}(Q) / s$, we can absorb the term $sc_3 I$ in (22) into Q and rewrite the constraints (21) and (22) as:

$$\left(\frac{G_{ij}(x) + G_{ji}(x)}{2} \right)^T P + P \left(\frac{G_{ij}(x) + G_{ji}(x)}{2} \right) + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 \le Q, \quad Q > 0, \ i < j, \ \text{s.t. } h_i \cap h_j \neq \phi \tag{23}$$

$$G_{ii}^T(x) P + P G_{ii}(x) + \frac{\gamma_f^2}{\sigma_1} I + \sigma_1 P^2 + (s-1) Q \le 0$$
$$i = 1, 2, \ldots, r \tag{24}$$

The LMIs (17) and (18) are obtained from these two constraints. First consider the constraint (24). Substituting (7) in (24) and multiplying it on the left and write by $X = P^{-1}$, we have

$$\Big(XA_i^T + A_iX - M_i^T B_i^T - B_iM_i + (s-1)Y_0 + \sigma_1 I + \frac{\gamma_f^2}{\sigma_1} XX\Big) + \Big(-M_i^T \delta_g^T(x) - \delta_g(x)M_i\Big) \le 0 \tag{25}$$

where $Y_0 = XQX > 0$ and $M_i = F_iX$. For the second term in (25) and using (4), with any positive constants σ_2 we have:

$$-M_i^T\delta_g^T(x) - \delta_g(x)M_i = -(\sigma_2 I + \delta_g(x)M_i)^T\left(I + \frac{1}{\sigma_2}\delta_g(x)M_i\right)$$
$$+\sigma_2 I + \frac{1}{\sigma_2}M_i^T\delta_g^T(x)\delta_g(x)M_i \le \sigma_2 I + \frac{1}{\sigma_2}M_i^T\delta_g^T(x)\delta_g(x)M_i$$
$$\le \sigma_2 I + \frac{\beta_g^2}{\sigma_2}M_i^T M_i$$

Substituting this inequality in (25), leads to the final constraint

$$XA_i^T + A_iX - M_i^T B_i^T - B_iM_i + (s-1)Y_0 + \sigma_1 I + \sigma_2 I + \frac{\gamma_f^2}{\sigma_1}XX + \frac{\beta_g^2}{\sigma_2}M_i^T M_i \le 0 \tag{26}$$

Applying the schur complement ([1]) to (26), transforms it into the LMI (17). The LMI (18) is obtained quite similarly from (23). Also, It can be easily seen through the proving procedure that $\kappa_f = 0$ and the corresponding row and column in (17) and (18) are dropped, when $\delta_f(x) = 0$. When $\delta_g(x) = 0$, the same holds for κ_g and the corresponding rows and columns in (17) and (18). Furthermore, since the analysis is valid globally, it is obvious that if the modeling (3) is valid globally, and the assumptions (4) hold globally too, then the origin is globally exponentially stable.

It can be seen that the constraints (16)-(18), which we found here for exponential stability, are not significantly more restrictive than similar results in the literature for asymptotic stability, cf. the results in [2].

IV. Examples

A. Stabilizing a Pendulum with Parametric Uncertainties

Consider the simple pendulum shown in Fig. 1. The dynamic equation of this pendulum, taken from [8], can be written as

$$ml\ddot{\theta} + mg\sin\theta + kl\dot{\theta} = \frac{T}{l} \tag{27}$$

where l is the length of the rod, m is the mass of the bob, k is the coefficient of friction, $g = 9.81\,(m/s^2)$ is the acceleration due to gravity, and T is the input torque applied to the pendulum. Suppose that, because of some practical reasons, there are some uncertainties in the physical quantities of the system, as they take the values: $0.97 \le l \le 1.03\,(m)$, $0.9 \le m \le 1.1\,(kg)$, and $0.05 \le k \le 0.15$. Considering the state variables as $x_1 = \theta - \varphi$ and $x_2 = \dot{\theta}$,

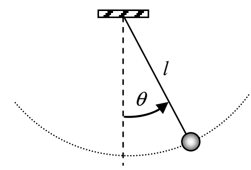

Fig. 1. A simple Pendulum.

we can write the state equations as

$$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin(x_1+\varphi) - bx_2 + cu\end{cases}$$

where $a = \frac{g}{l}$, $b = \frac{k}{m}$, $c = \frac{1}{ml^2}$, and $u = T$. We can represent this system as

$$\dot{x} = \hat{f}(x) + \hat{g}(x)u + \delta_f(x) + \delta_g(x)u \tag{28}$$

where

$$\hat{f}(x) = \begin{bmatrix} x_2 \\ -\hat{a}\sin(x_1+\varphi) - \hat{b}x_2\end{bmatrix}, \qquad \hat{g}(x) = \begin{bmatrix}0\\ \hat{c}\end{bmatrix}$$

$$\delta_f(x) = (\hat{a}-a)\sin(x_1+\varphi) + (\hat{b}-b)x_2, \qquad \delta_g(x) = c - \hat{c}$$

and

$$\hat{a} = (a_{\max}+a_{\min})/2, \hat{b} = (b_{\max}+b_{\min})/2, \hat{c} = (c_{\max}+c_{\min})/2$$

Substituting the physical values, we have: $\hat{a} = 9.82$, $\hat{b} = 0.1$, and $\hat{c} = 1.02$. Employing the sector nonlinearity approach found in [2] or [3], the nominal system can be globally and exactly represented, using only two fuzzy rules as following:

$$\dot{x} = \hat{f}(x) + \hat{g}(x)u = \sum_{i=1}^{2} h_i(x)(A_ix + B_iu) \tag{29}$$

For $\varphi = \pi$, which corresponds to the unstable upright equilibrium position, we have

$$h_1(x) = \frac{1}{11.95}\left(9.82\frac{\sin x}{x} + 2.13\right)$$

$$h_2(x) = \frac{1}{11.95}\left(9.82 - 9.82\frac{\sin x}{x}\right)$$

$$A_1 = \begin{bmatrix}0 & 1\\ 9.82 & -0.2\end{bmatrix}, A_2 = \begin{bmatrix}0 & 1\\ -2.13 & -0.2\end{bmatrix}, B_1 = B_2 = \begin{bmatrix}0\\ 1.02\end{bmatrix}$$

Since the modeling is exact, the perturbation terms originates only from the system uncertainties, and can be easily shown that they globally satisfy (4) with $\gamma_f = 0.3$, and $\beta_g = 0.16$. Solving the LMIs (16)-(18) for this system, we obtain the PDC feedback gains as $F_1 = [131.7,\ 39.9]$ and $F_2 = [199.3,\ 61.9]$.

Although the modeling is valid globally and the constraints (4) are also satisfied globally, we can not say that the controller globally stabilizes the unstable equilibrium point, because the system does not have a unique equilibrium point. However from the physical point of view, the pendulum is stabilized at its upright position, starting from any initial positions and velocities, except from

$\theta(0)=0, \dot{\theta}(0)=0$. Simulation results considering $m=1.1$, $l=1.03$, and $k=0.05$, are shown in Fig. 2 for zero initial angular velocity and different initial angles.

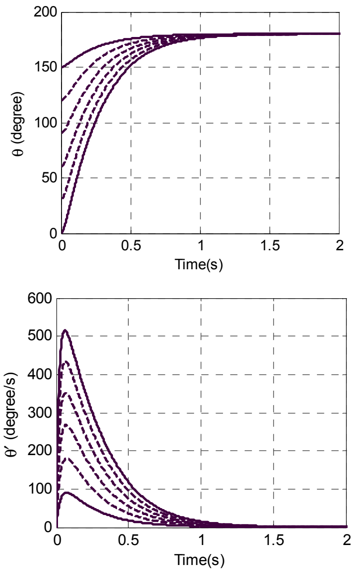

Fig. 2. Angle and angular velocity of the pendulum, starting from different initial angles and zero angular velocity

B. Stability in the Presence of Modeling Error

Consider the following system taken from [3]:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 \cos x_2 - x_3 \\ \dot{x}_3 = x_1 x_3 + (1+\varepsilon \sin x_3) u \end{cases} \tag{30}$$

By assuming $-5 < x_1 < 5$, $-\pi/2 < x_2 < \pi/2$, $-\pi < x_3 < \pi$, and $\varepsilon = 0.5$, this system can be locally, but exactly represented using eight fuzzy rules, as done in [3]. However, to reduce the number of fuzzy rules, we can write (30) as $\dot{x} = \hat{f}(x) + \hat{g}(x)u + \delta_f(x)$ where

$$\hat{f}(x) = \begin{pmatrix} x_2 \\ 0.5x_1 - x_3 \\ x_1 x_3 \end{pmatrix}, \hat{g}(x) = \begin{pmatrix} 0 \\ 0 \\ 1+\varepsilon \sin x_3 \end{pmatrix}$$

$$\delta_f(x) = \begin{pmatrix} 0 \\ x_1(\cos x_2 - 0.5) \\ 0 \end{pmatrix}$$

Now, the nominal system can be exactly represented using four rules, with the same weights and matrices as given in [3] in the reduction with respect to $A(2,1)$, and due to space limitation we avoid repeating them here. In this example, the perturbation term originates only from the modeling error and it can be easily shown that it satisfies (4) with $\gamma_f = 0.5$, and $\beta_g = 0$. Solving the LMIs (16)-(18) for this system, we obtain the PDC feedback gains:

$F_1 = [-133.8, -145.4, 28.4]$, $F_2 = [-150.7, -163.7, 24.5]$
$F_3 = [-451.6, -490.2, 92.7]$, $F_4 = [-603.8, -653.3, 109.9]$

Fig. 3 shows the simulation results obtained when we apply the designed PDC to the original nonlinear system.

Since the modeling is valid locally, the stability analysis is also valid locally and we can not say any statements about global stability.

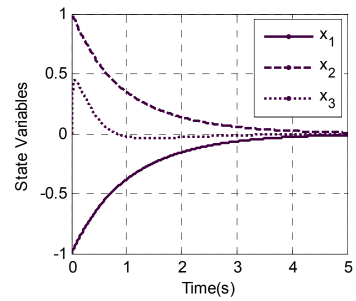

Fig. 3. State variables of the closed loop system of example B.

V. CONCLUSION

In this paper, we proposed a fuzzy-model-based approach for exponentially stabilizing perturbed nonlinear systems. We also considered the modeling error and presented it uniformly as system perturbations. We showed the possibility of this presentation for affine nonlinear systems, and finally, we provide the design constraints in the form of LMIs. We should also notify here that the approach described in this paper was independent of the method used to find the TS fuzzy approximation. So, it can be potentially used for other methods which were not considered in this paper or will be developed later.

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